

Generalized Uncertainty Relation Associated with a Monotone or an Anti-Monotone Pair Skew Information

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Abstract. We give a trace inequality related to the uncertainty relation based on the monotone or anti-monotone pair skew information which is one of generalizations of result given by [6]. And it includes the result for generalized Wigner-Yanase-Dyson skew information as a particular case ([14]).

Key Words: Uncertainty relation, Wigner-Yanase-Dyson skew information

1 Introduction

Wigner-Yanase skew information

$$\begin{aligned} I_{\rho}(H) &= \frac{1}{2} \text{Tr} \left[(i [\rho^{1/2}, H])^2 \right] \\ &= \text{Tr}[\rho H^2] - \text{Tr}[\rho^{1/2} H \rho^{1/2} H] \end{aligned}$$

was defined in [11]. This quantity can be considered as a kind of the degree for non-commutativity between a quantum state ρ and an observable H . Here we denote the commutator by $[X, Y] = XY - YX$. This quantity was generalized by Dyson

$$\begin{aligned} I_{\rho, \alpha}(H) &= \frac{1}{2} \text{Tr}[(i[\rho^{\alpha}, H])(i[\rho^{1-\alpha}, H])] \\ &= \text{Tr}[\rho H^2] - \text{Tr}[\rho^{\alpha} H \rho^{1-\alpha} H], \alpha \in [0, 1] \end{aligned}$$

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which is known as the Wigner-Yanase-Dyson skew information. It is famous that the convexity of $I_{\rho,\alpha}(H)$ with respect to ρ was successfully proven by E.H.Lieb in [8]. And also this quantity was generalized by Cai and Luo

$$\begin{aligned} & I_{\rho,\alpha,\beta}(H) \\ &= \frac{1}{2} \text{Tr}[(i[\rho^\alpha, H])(i[\rho^\beta, H])\rho^{1-\alpha-\beta}] \\ &= \frac{1}{2} \{ \text{Tr}[\rho H^2] + \text{Tr}[\rho^{\alpha+\beta} H \rho^{1-\alpha-\beta} H] - \text{Tr}[\rho^\alpha H \rho^{1-\alpha} H] - \text{Tr}[\rho^\beta H \rho^{1-\beta} H] \}, \end{aligned}$$

where $\alpha, \beta \geq 0, \alpha + \beta \leq 1$. The convexity of $I_{\rho,\alpha,\beta}(H)$ with respect to ρ was proven by Cai and Luo in [2] under some restrictive condition. In this paper we let $M_n(\mathbb{C})$ be the set of all $n \times n$ complex matrices, $M_{n,sa}(\mathbb{C})$ be the set of all $n \times n$ self-adjoint matrices, $M_{n,+}(\mathbb{C})$ be the set of strictly positive elements of $M_n(\mathbb{C})$ and $M_{n,+,1}(\mathbb{C})$ be the set of strictly positive density matrices, that is $M_{n,+,1}(\mathbb{C}) = \{\rho \in M_n(\mathbb{C}) | \text{Tr}[\rho] = 1, \rho > 0\}$. If it is not otherwise specified, from now on we shall treat the case of faithful states, that is $\rho > 0$. The relation between the Wigner-Yanase skew information and the uncertainty relation was studied in [10]. Moreover the relation between the Wigner-Yanase-Dyson skew information and the uncertainty relation was studied in [7, 12]. In our paper [12] and [13], we defined a generalized skew information and then derived a kind of an uncertainty relations. And also in [14] and [15], we gave an uncertainty relation of two parameter generalized Wigner-Yanase-Dyson skew information. In this paper, we consider three parameter generalized Wigner-Yanase-Dyson skew information and give a kind of generalized uncertainty relations which is a generalization of the result of Ko and Yoo [6].

2 Trace inequality of Wigner-Yanase-Dyson skew information

We review the relation between the Wigner-Yanase skew information and the uncertainty relation. In quantum mechanical system, the expectation value of an observable H in a quantum state ρ is expressed by $\text{Tr}[\rho H]$. It is natural that the variance for a quantum state ρ and an observable H is defined by $V_\rho(H) = \text{Tr}[\rho(H - \text{Tr}[\rho H]I)^2] = \text{Tr}[\rho H^2] - \text{Tr}[\rho H]^2$. It is famous that we have

$$V_\rho(A)V_\rho(B) \geq \frac{1}{4} |\text{Tr}[\rho[A, B]]|^2 \quad (2.1)$$

for a quantum state ρ and two observables A and B . The further strong results was given by Schrödinger

$$V_\rho(A)V_\rho(B) - |\text{Re}\{\text{Cov}_\rho(A, B)\}|^2 \geq \frac{1}{4} |\text{Tr}[\rho[A, B]]|^2,$$

where the covariance is defined by $Cov_\rho(A, B) = Tr[\rho(A - Tr[\rho A]I)(B - Tr[\rho B]I)]$. However, the uncertainty relation for the Wigner-Yanase skew information failed. (See [10, 7, 12])

$$I_\rho(A)I_\rho(B) \geq \frac{1}{4}|Tr[\rho[A, B]]|^2.$$

Recently, S.Luo introduced the quantity $U_\rho(H)$ representing a quantum uncertainty excluding the classical mixture:

$$U_\rho(H) = \sqrt{V_\rho(H)^2 - (V_\rho(H) - I_\rho(H))^2}, \quad (2.2)$$

then he derived the uncertainty relation on $U_\rho(H)$ in [9]:

$$U_\rho(A)U_\rho(B) \geq \frac{1}{4}|Tr[\rho[A, B]]|^2. \quad (2.3)$$

Note that we have the following relation

$$0 \leq I_\rho(H) \leq U_\rho(H) \leq V_\rho(H). \quad (2.4)$$

The inequality (2.3) is a refinement of the inequality (2.1) in the sense of (2.4). In [13], we studied one-parameter extended inequality for the inequality (2.3).

Definition 2.1 For $0 \leq \alpha \leq 1$, a quantum state ρ and an observable H , we define the Wigner-Yanase-Dyson skew information

$$\begin{aligned} I_{\rho, \alpha}(H) &= \frac{1}{2}Tr[(i[\rho^\alpha, H_0])(i[\rho^{1-\alpha}, H_0])] \\ &= Tr[\rho H_0^2] - Tr[\rho^\alpha H_0 \rho^{1-\alpha} H_0] \end{aligned}$$

and we also define

$$\begin{aligned} J_{\rho, \alpha}(H) &= \frac{1}{2}Tr[\{\rho^\alpha, H_0\}\{\rho^{1-\alpha}, H_0\}] \\ &= Tr[\rho H_0^2] + Tr[\rho^\alpha H_0 \rho^{1-\alpha} H_0], \end{aligned}$$

where $H_0 = H - Tr[\rho H]I$ and we denote the anti-commutator by $\{X, Y\} = XY + YX$.

Note that we have

$$\frac{1}{2}Tr[(i[\rho^\alpha, H_0])(i[\rho^{1-\alpha}, H_0])] = \frac{1}{2}Tr[(i[\rho^\alpha, H])(i[\rho^{1-\alpha}, H])]$$

but we have

$$\frac{1}{2}Tr[\{\rho^\alpha, H_0\}\{\rho^{1-\alpha}, H_0\}] \neq \frac{1}{2}Tr[\{\rho^\alpha, H\}\{\rho^{1-\alpha}, H\}].$$

Then we have the following inequalities:

$$I_{\rho,\alpha}(H) \leq I_\rho(H) \leq J_\rho(H) \leq J_{\rho,\alpha}(H), \quad (2.5)$$

since we have $Tr[\rho^{1/2}H\rho^{1/2}H] \leq Tr[\rho^\alpha H\rho^{1-\alpha}H]$. (See [1, 3] for example.) If we define

$$U_{\rho,\alpha}(H) = \sqrt{V_\rho(H)^2 - (V_\rho(H) - I_{\rho,\alpha}(H))^2}, \quad (2.6)$$

as a direct generalization of Eq.(2.2), then we have

$$0 \leq I_{\rho,\alpha}(H) \leq U_{\rho,\alpha}(H) \leq U_\rho(H) \quad (2.7)$$

due to the first inequality of (2.5). We also have

$$U_{\rho,\alpha}(H) = \sqrt{I_{\rho,\alpha}(H)J_{\rho,\alpha}(H)}.$$

From the inequalities (2.4),(2.6),(2.7), our situation is that we have

$$0 \leq I_{\rho,\alpha}(H) \leq I_\rho(H) \leq U_\rho(H)$$

and

$$0 \leq I_{\rho,\alpha}(H) \leq U_{\rho,\alpha}(H) \leq U_\rho(H).$$

We gave the following uncertainty relation with respect to $U_{\rho,\alpha}(H)$ as a direct generalization of the inequality (2.3).

Theorem 2.1 ([13]) *For $0 \leq \alpha \leq 1$, a quantum state ρ and observables A, B ,*

$$U_{\rho,\alpha}(A)U_{\rho,\alpha}(B) \geq \alpha(1-\alpha)|Tr[\rho[A, B]]|^2. \quad (2.8)$$

Now we define the two parameter extensions of Wigner-Yanase skew information and give an uncertainty relation under some conditions.

Definition 2.2 *For $\alpha, \beta \geq 0$, a quantum state ρ and an observable H , we define the generalized Wigner-Yanase-Dyson skew information*

$$\begin{aligned} & I_{\rho,\alpha,\beta}(H) \\ &= \frac{1}{2}Tr [(i[\rho^\alpha, H_0])(i[\rho^\beta, H_0])\rho^{1-\alpha-\beta}] \\ &= \frac{1}{2}\{Tr[\rho H_0^2] + Tr[\rho^{\alpha+\beta}H_0\rho^{1-\alpha-\beta}H_0] - Tr[\rho^\alpha H_0\rho^{1-\alpha}H_0] - Tr[\rho^\beta H_0\rho^{1-\beta}H_0]\} \end{aligned}$$

and we define

$$\begin{aligned}
& J_{\rho,\alpha,\beta}(H) \\
&= \frac{1}{2} \text{Tr} [\{\rho^\alpha, H_0\} \{\rho^\beta, H_0\} \rho^{1-\alpha-\beta}] \\
&= \frac{1}{2} \{ \text{Tr}[\rho H_0^2] + \text{Tr}[\rho^{\alpha+\beta} H_0 \rho^{1-\alpha-\beta} H_0] + \text{Tr}[\rho^\alpha H_0 \rho^{1-\alpha} H_0] + \text{Tr}[\rho^\beta H_0 \rho^{1-\beta} H_0] \},
\end{aligned}$$

where $H_0 = H - \text{Tr}[\rho H]I$ and we denote the anti-commutator by $\{X, Y\} = XY + YX$. We remark that $\alpha + \beta = 1$ implies $I_{\rho,\alpha}(H) = I_{\rho,\alpha,1-\alpha}(H)$ and $J_{\rho,\alpha}(H) = J_{\rho,\alpha,1-\alpha}(H)$. We also define

$$U_{\rho,\alpha,\beta}(H) = \sqrt{I_{\rho,\alpha,\beta}(H) J_{\rho,\alpha,\beta}(H)}.$$

In this paper we assume that $\alpha, \beta \geq 0$ do not necessarily satisfy the condition $\alpha + \beta \leq 1$. We give the following theorem.

Theorem 2.2 ([14]) *For $\alpha, \beta \geq 0$ and $\alpha + \beta \geq 1$ or $\alpha + \beta \leq \frac{1}{2}$ and observables A, B ,*

$$U_{\rho,\alpha,\beta}(A) U_{\rho,\alpha,\beta}(B) \geq \alpha\beta |\text{Tr}[\rho[A, B]]|^2. \quad (2.9)$$

And we also define the two parameter extensions of Wigner-Yanase skew information which are different from Definition 2.2.

Definition 2.3 *For $\alpha, \beta \geq 0$, a quantum state ρ and an observable H , we define the generalized Wigner-Yanase-Dyson skew information*

$$\begin{aligned}
& \tilde{I}_{\rho,\alpha,\beta}(H) \\
&= \frac{1}{2} \text{Tr} [(i[\rho^\alpha, H_0])(i[\rho^\beta, H_0])] \\
&= \text{Tr}[\rho^{\alpha+\beta} H_0^2] - \text{Tr}[\rho^\alpha H_0 \rho^\beta H_0].
\end{aligned}$$

and we define

$$\begin{aligned}
& \tilde{J}_{\rho,\alpha,\beta}(H) \\
&= \frac{1}{2} \text{Tr} [\{\rho^\alpha, H_0\} \{\rho^\beta, H_0\}] \\
&= \text{Tr}[\rho^{\alpha+\beta} H_0^2] + \text{Tr}[\rho^\alpha H_0 \rho^\beta H_0],
\end{aligned}$$

where $H_0 = H - \text{Tr}[\rho H]I$ and we denote the anti-commutator by $\{X, Y\} = XY + YX$. We remark that $\alpha + \beta = 1$ implies $I_{\rho,\alpha}(H) = \tilde{I}_{\rho,\alpha,1-\alpha}(H)$ and $J_{\rho,\alpha}(H) = \tilde{J}_{\rho,\alpha,1-\alpha}(H)$. We also define

$$\tilde{U}_{\rho,\alpha,\beta}(H) = \sqrt{\tilde{I}_{\rho,\alpha,\beta}(H) \tilde{J}_{\rho,\alpha,\beta}(H)}.$$

Then we give the following theorem.

Theorem 2.3 ([15]) *For $\alpha, \beta \geq 0$ ($\alpha\beta \neq 0$) and observables A, B ,*

$$\tilde{U}_{\rho, \alpha, \beta}(A)\tilde{U}_{\rho, \alpha, \beta}(B) \geq \frac{\alpha\beta}{(\alpha + \beta)^2} |Tr[\rho^{\alpha+\beta}[A, B]]|^2.$$

Remark 2.1 *We remark that (2.8) is derived by putting $\beta = 1 - \alpha$ in (2.9). Then Theorem 2.2 is a generalization of Theorem 2.1 given in [13].*

3 Trace inequality of monotone or anti-monotone pair skew information

Definition 3.1 *Let $f(x), g(x)$ be nonnegative continuous functions defined on the interval $[0, 1]$. We call the pair (f, g) a compatible in log-increase, monotone pair (CLI monotone pair, in short) if*

- (a) $(f(x) - f(y))(g(x) - g(y)) \geq 0$ for all $x, y \in [0, 1]$.
- (b) $f(x), g(x)$ are differentiable on $(0, 1)$ and

$$0 \leq \inf_{0 < x < 1} \frac{G'(x)}{F'(x)} \leq \sup_{0 < x < 1} \frac{G'(x)}{F'(x)} < \infty,$$

where $F(x) = \log f(x), G(x) = \log g(x)$.

Definition 3.2 *Let $f(x), g(x)$ be nonnegative continuous functions defined on the interval $[0, 1]$. We call the pair (f, g) a compatible in log-increase, anti-monotone pair (CLI anti-monotone pair, in short) if*

- (a) $(f(x) - f(y))(g(x) - g(y)) \leq 0$ for all $x, y \in [0, 1]$.
- (b) $f(x), g(x)$ are differentiable on $(0, 1)$ and

$$-\infty < \inf_{0 < x < 1} \frac{G'(x)}{F'(x)} \leq \sup_{0 < x < 1} \frac{G'(x)}{F'(x)} \leq 0,$$

where $F(x) = \log f(x), G(x) = \log g(x)$.

Let $f(x), g(x), h(x)$ be nonnegative continuous functions defined on $[0, 1]$ and be differentiable on $(0, 1)$. We assume that (f, g) is CLI monotone pair and (f, h) is CLI monotone or anti-monotone pair. We introduce the correlation functions in the following way.

Definition 3.3

$$\begin{aligned}
I_{\rho,(f,g,h)}(H) &= \frac{1}{2} \text{Tr}[(i[f(\rho), H_0])(i[g(\rho), H_0])h(\rho)] \\
&= -\frac{1}{2} \text{Tr}[(f(\rho), H_0)([g(\rho), H_0])h(\rho)] \\
&= -\frac{1}{2} \text{Tr}[(f(\rho)H_0 - H_0f(\rho))(g(\rho)H_0 - H_0g(\rho))h(\rho)] \\
&= -\frac{1}{2} \text{Tr}[f(\rho)H_0g(\rho)H_0h(\rho) - f(\rho)H_0^2g(\rho)h(\rho)] \\
&\quad + \frac{1}{2} \text{Tr}[H_0f(\rho)g(\rho)H_0h(\rho) - H_0f(\rho)H_0g(\rho)h(\rho)] \\
&= -\frac{1}{2} \text{Tr}[f(\rho)h(\rho)H_0g(\rho)H_0 - f(\rho)g(\rho)h(\rho)H_0^2] \\
&\quad + \frac{1}{2} \text{Tr}[f(\rho)g(\rho)H_0h(\rho)H_0 - g(\rho)h(\rho)H_0f(\rho)H_0] \\
&= \frac{1}{2} \{ \text{Tr}[f(\rho)g(\rho)h(\rho)H_0^2] + \text{Tr}[f(\rho)g(\rho)H_0h(\rho)H_0] \} \\
&\quad - \frac{1}{2} \{ \text{Tr}[f(\rho)H_0g(\rho)h(\rho)H_0] + \text{Tr}[g(\rho)H_0f(\rho)h(\rho)H_0] \}.
\end{aligned}$$

$$\begin{aligned}
J_{\rho,(f,g,h)}(H) &= \frac{1}{2} \text{Tr}[\{f(\rho), H_0\}\{g(\rho), H_0\}h(\rho)] \\
&= \frac{1}{2} \text{Tr}[(f(\rho)H_0 + H_0f(\rho))(g(\rho)H_0 + H_0g(\rho))h(\rho)] \\
&= \frac{1}{2} \text{Tr}[f(\rho)H_0g(\rho)H_0h(\rho) + f(\rho)H_0^2g(\rho)h(\rho)] \\
&\quad + \frac{1}{2} \text{Tr}[H_0f(\rho)g(\rho)H_0h(\rho) + H_0f(\rho)H_0g(\rho)h(\rho)] \\
&= \frac{1}{2} \{ \text{Tr}[f(\rho)g(\rho)h(\rho)H_0^2] + \text{Tr}[f(\rho)g(\rho)H_0h(\rho)H_0] \} \\
&\quad + \frac{1}{2} \{ \text{Tr}[f(\rho)H_0g(\rho)h(\rho)H_0] + \text{Tr}[g(\rho)H_0f(\rho)h(\rho)H_0] \}.
\end{aligned}$$

$$U_{\rho,(f,g,h)}(H) = \sqrt{I_{\rho,(f,g,h)}(H)J_{\rho,(f,g,h)}(H)}.$$

We are ready to state our main result. For f, g, h we let

$$\begin{aligned}
&\beta(f, g, h) \\
&= \min\left\{\frac{m}{(1+m+n)^2}, \frac{m}{(1+m+N)^2}, \frac{M}{(1+M+n)^2}, \frac{M}{(1+M+N)^2}\right\}, \quad (3.1)
\end{aligned}$$

where

$$m = \inf_{0 < x < 1} \frac{G'(x)}{F'(x)}, \quad M = \sup_{0 < x < 1} \frac{G'(x)}{F'(x)}$$

$$n = \inf_{0 < x < 1} \frac{H'(x)}{F'(x)}, \quad N = \sup_{0 < x < 1} \frac{H'(x)}{F'(x)}.$$

We consider the following two assumptions.

(I) $(f, g), (f, h)$ are CLI monotone pair satisfying

$$1 + \frac{G(y) - G(x)}{F(y) - F(x)} \leq \frac{H(y) - H(x)}{F(y) - F(x)} \quad \text{for } x < y,$$

where $F(x) = \log f(x)$, $G(x) = \log g(x)$, $H(x) = \log h(x)$

(II) (f, g) is CLI monotone pair and (f, h) is CLI anti-monotone pair satisfying

$$1 + \frac{G(y) - G(x)}{F(y) - F(x)} + \frac{H(y) - H(x)}{F(y) - F(x)} \geq 0 \quad \text{for } x < y.$$

Theorem 3.1 *Under the assumption (I) or (II), the following inequality holds:*

$$U_{\rho, (f, g, h)}(A)U_{\rho, (f, g, h)}(B) \geq \beta(f, g, h)|\text{Tr}[f(\rho)g(\rho)h(\rho)[A, B]]|^2$$

for $A, B \in M_{n, sa}(\mathbb{C})$.

4 Proof of Theorem 3.1

Let $\rho = \sum_{i=1}^n \lambda_i |\phi_i\rangle\langle\phi_i| \in M_{n, +, 1}(\mathbb{C})$, where $\{|\phi_i\rangle\}_{i=1}^n$ is an orthonormal set in \mathbb{C}^n . Let (f, g) be a CLI monotone pair and (f, h) be a CLI monotone or anti-monotone pair. By a simple calculation, we have for any $H \in M_{n, sa}(\mathbb{C})$

$$\text{Tr}[f(\rho)g(\rho)h(\rho)H_0^2] = \sum_{i,j} \frac{1}{2} \{f(\lambda_i)g(\lambda_i)h(\lambda_i) + f(\lambda_j)g(\lambda_j)h(\lambda_j)\} |a_{ij}|^2. \quad (4.1)$$

$$\text{Tr}[f(\rho)g(\rho)H_0h(\rho)H_0] = \sum_{i,j} \frac{1}{2} \{f(\lambda_i)g(\lambda_i)h(\lambda_j) + f(\lambda_j)g(\lambda_j)h(\lambda_i)\} |a_{ij}|^2. \quad (4.2)$$

$$\text{Tr}[f(\rho)H_0g(\rho)h(\rho)H_0] = \sum_{i,j} \frac{1}{2} \{f(\lambda_i)g(\lambda_j)h(\lambda_j) + f(\lambda_j)g(\lambda_i)h(\lambda_i)\} |a_{ij}|^2. \quad (4.3)$$

$$\text{Tr}[g(\rho)H_0f(\rho)h(\rho)H_0] = \sum_{i,j} \frac{1}{2} \{g(\lambda_i)f(\lambda_j)h(\lambda_j) + g(\lambda_j)f(\lambda_i)h(\lambda_i)\} |a_{ij}|^2, \quad (4.4)$$

where $a_{ij} = \langle\phi_i|H_0|\phi_j\rangle$ and $a_{ij} = \overline{a_{ji}}$. From (4.1) - (4.4), we get

$$I_{\rho, (f, g, h)}(H) = \frac{1}{2} \sum_{i < j} (f(\lambda_i) - f(\lambda_j))(g(\lambda_i) - g(\lambda_j))(h(\lambda_i) + h(\lambda_j)) |a_{ij}|^2.$$

$$J_{\rho,(f,g,h)}(H) \geq \frac{1}{2} \sum_{i < j} (f(\lambda_i) + f(\lambda_j))(g(\lambda_i) + g(\lambda_j))(h(\lambda_i) + h(\lambda_j))|a_{ij}|^2.$$

To prove Theorem 3.1, we need to control a lower bound of a functional coming from a CLI monotone or anti-monotone pair. For f, g, h satisfying (I) or (II), we define a function L on $[0, 1] \times [0, 1]$ by

$$L(x, y) = \frac{(f(x)^2 - f(y)^2)(g(x)^2 - g(y)^2)(h(x) + h(y))^2}{(f(x)g(x)h(x) - f(y)g(y)h(y))^2}. \quad (4.5)$$

Proposition 4.1 *Under the assumption (I) or (II)*

$$\min_{x, y \in [0, 1]} L(x, y) \geq 16\beta(f, g, h),$$

where $\beta(f, g, h)$ is defined in (3.1).

For the proof of Proposition 4.1, we need the following lemma.

Lemma 4.1 *If $a, b, c \geq 0$ satisfy $0 < a + b \leq c$ or if $a, b \geq 0, c \leq 0$ satisfy $a + b + c > 0$, then the inequality*

$$\frac{(e^{2ar} - 1)(e^{2br} - 1)(e^{cr} + 1)^2}{(e^{(a+b+c)r} - 1)^2} \geq \frac{16ab}{(a + b + c)^2}$$

holds for any real number r .

Proof. We put $e^r = t$. Then we may prove the following;

$$(t^{2a} - 1)(t^{2b} - 1)(t^c + 1)^2 \geq \frac{16ab}{(a + b + c)^2}(t^{a+b+c} - 1)^2 \quad (4.6)$$

for $t > 0$. It is sufficient to prove (4.6) for $t \geq 1$ and $a, b, c \geq 0, 0 < a + b \leq c$ or $a, b \geq 0, c \leq 0, a + b + c > 0$.

By Lemma 3.3 in [13] we have for $0 \leq p \leq 1$ and $s \geq 1$,

$$(s^{2p} - 1)(s^{2(1-p)} - 1) \geq 4p(1-p)(s - 1)^2.$$

We assume that $a, b \geq 0$. We put $p = a/(a + b)$ and $s^{1/(a+b)} = t$. Then

$$(t^{2a} - 1)(t^{2b} - 1) \geq \frac{4ab}{(a + b)^2}(t^{a+b} - 1)^2.$$

Then we have

$$(t^{2a} - 1)(t^{2b} - 1)(t^c + 1)^2 \geq \frac{4ab}{(a + b)^2}(t^{a+b} - 1)^2(t^c + 1)^2.$$

In order to show the aimed inequality, we have to prove that

$$(t^{a+b} - 1)^2(t^c + 1)^2 \geq \frac{4(a+b)^2}{(a+b+c)^2}(t^{a+b+c} - 1)^2.$$

Since $a+b+c > 0$, it is sufficient to prove the following inequality

$$(t^{a+b} - 1)(t^c + 1) \geq \frac{2(a+b)}{a+b+c}(t^{a+b+c} - 1) \quad (4.7)$$

for $t \geq 1$ and $a, b, c \geq 0, 0 < a+b \leq c$ or $a, b \geq 0, c \leq 0, a+b+c > 0$. We put

$$S(t) = (t^{a+b} - 1)(t^c + 1) - \frac{2(a+b)}{a+b+c}(t^{a+b+c} - 1).$$

Then

$$S'(t) = t^{c-1}\{(c-a-b)t^{a+b} - c + (a+b)t^{a+b-c}\}.$$

Here we put

$$T(t) = (c-a-b)t^{a+b} - c + (a+b)t^{a+b-c}.$$

Then

$$T'(t) = (a+b)(c-a-b)t^{a+b-c-1}(t^c - 1).$$

When $a+b \leq c$, $T'(t) \geq 0$. Since $T(1) = 0$, $T(t) \geq 0$ for $t \geq 1$. Then $S'(t) \geq 0$. Since $S(1) = 0$, $S(t) \geq 0$ for $t \geq 1$. On the other hand when $c \leq 0$, $T'(t) \geq 0$. Since $T(1) = 0$, $T(t) \geq 0$ for $t \geq 1$. Then $S'(t) \geq 0$. Since $S(1) = 0$, $S(t) \geq 0$ for $t \geq 1$. Hence we get (4.7). \square

Proof of Proposition 4.1. Let $x < y$. In the last line of (4.5), dividing both the numerator and the denominator by $(f(x)g(x)h(x))^2$ and by using $F(x) = \log f(x)$, $G(x) = \log g(x)$ and $H(x) = \log h(x)$, we get

$$L(x, y) = \frac{(e^{2(F(y)-F(x))} - 1)(e^{2(G(y)-G(x))} - 1)(e^{H(y)-H(x)} + 1)^2}{(e^{F(y)-F(x)+G(y)-G(x)+H(y)-H(x)} - 1)^2}$$

By the generalized mean value theorem, there exist z ($x < z < y$), w ($x < w < y$) such that

$$\frac{G(y) - G(x)}{F(y) - F(x)} = \frac{G'(z)}{F'(z)} = k(z), \quad \frac{H(y) - H(x)}{F(y) - F(x)} = \frac{H'(w)}{F'(w)} = \ell(w).$$

Thus we have

$$L(x, y) = \frac{(e^{2(F(y)-F(x))} - 1)(e^{2k(z)(F(y)-F(x))} - 1)(e^{\ell(w)(F(y)-F(x))} + 1)^2}{(e^{(1+k(z)+\ell(w))(F(y)-F(x))} - 1)^2}.$$

It follows from Lemma 4.1 that for any $R > 0$, the function

$$(k, \ell) \rightarrow A(k, \ell) = \frac{(R^2 - 1)(R^{2k} - 1)(R^\ell + 1)^2}{(R^{(1+k+\ell)} - 1)^2}$$

defined in $k \in [m, M], \ell \in [n, N]$ is bounded from below by $\min_{m \leq k \leq M, n \leq \ell \leq N} \{A(k, \ell)\}$. It is easy to obtain

$$\min_{m \leq k \leq M, n \leq \ell \leq N} \{A(k, \ell)\} \geq 16\beta(f, g, h).$$

We complete the proof. \square

Proof of Theorem 3.1. Since

$$\begin{aligned} \text{Tr}[f(\rho)g(\rho)h(\rho)[A, B]] &= \text{Tr}[f(\rho)g(\rho)h(\rho)[A_0, B_0]] \\ &= 2i\text{Im}\{\text{Tr}[f(\rho)g(\rho)h(\rho)A_0B_0]\} \\ &= 2i\text{Im} \sum_{\ell < m} (f(\lambda_\ell)g(\lambda_\ell)h(\lambda_\ell) - f(\lambda_m)g(\lambda_m)h(\lambda_m))a_{m\ell}b_{\ell m} \\ &= 2i \sum_{\ell < m} (f(\lambda_\ell)g(\lambda_\ell)h(\lambda_\ell) - f(\lambda_m)g(\lambda_m)h(\lambda_m))\text{Im}(a_{m\ell}b_{\ell m}) \end{aligned}$$

for any $A, B \in M_{n,sa}(\mathbb{C})$, where $a_{\ell m} = \langle \phi_m | A_0 | \phi_\ell \rangle$ and $b_{m\ell} = \langle \phi_m | B_0 | \phi_m \rangle$, we have

$$\begin{aligned} |\text{Tr}[f(\rho)g(\rho)h(\rho)[A, B]]| &\leq 2 \sum_{\ell < m} |f(\lambda_\ell)g(\lambda_\ell)h(\lambda_\ell) - f(\lambda_m)g(\lambda_m)h(\lambda_m)| |\text{Im}a_{m\ell}b_{\ell m}| \\ &\leq 2 \sum_{\ell < m} |f(\lambda_\ell)g(\lambda_\ell)h(\lambda_\ell) - f(\lambda_m)g(\lambda_m)h(\lambda_m)| |a_{m\ell}| |b_{m\ell}|. \end{aligned}$$

By Proposition 4.1, we have

$$\begin{aligned} &\beta(f, g, h) |\text{Tr}[f(\rho)g(\rho)h(\rho)[A, B]]|^2 \\ &\leq 4\beta(f, g, h) \left(\sum_{\ell < m} |f(\lambda_\ell)g(\lambda_\ell)h(\lambda_\ell) - f(\lambda_m)g(\lambda_m)h(\lambda_m)| |a_{m\ell}| |b_{\ell m}| \right)^2 \\ &\leq \frac{1}{4} \left(\sum_{\ell < m} \sqrt{(f(\lambda_\ell)^2 - f(\lambda_m)^2)(g(\lambda_\ell)^2 - g(\lambda_m)^2)(h(\lambda_\ell) + h(\lambda_m))^2} |a_{\ell m}| |b_{m\ell}| \right)^2 \\ &= \frac{1}{4} \left(\sum_{\ell < m} \sqrt{\Delta_f(\ell, m)\Delta_g(\ell, m)\Gamma_h(\ell, m)} |a_{m\ell}| \sqrt{\Gamma_f(\ell, m)\Gamma_g(\ell, m)\Gamma_h(\ell, m)} |b_{\ell m}| \right)^2, \end{aligned}$$

where $\Delta_f(\ell, m) = f(\lambda_\ell) - f(\lambda_m)$, $\Delta_g(\ell, m) = g(\lambda_\ell) - g(\lambda_m)$ and $\Gamma_f(\ell, m) = f(\lambda_\ell) + f(\lambda_m)$, $\Gamma_g(\ell, m) = g(\lambda_\ell) + g(\lambda_m)$, $\Gamma_h(\ell, m) = h(\lambda_\ell) + h(\lambda_m)$. By Schwarz inequality, we have

$$\begin{aligned} &\beta(f, g, h) |\text{Tr}[f(\rho)g(\rho)h(\rho)[A, B]]|^2 \\ &\leq \frac{1}{2} \sum_{\ell < m} \Delta_f(\ell, m)\Delta_g(\ell, m)\Gamma_h(\ell, m) |a_{m\ell}|^2 \\ &\quad \times \frac{1}{2} \sum_{\ell < m} \Gamma_f(\ell, m)\Gamma_g(\ell, m)\Gamma_h(\ell, m) |b_{\ell m}|^2 \\ &\leq I_{\rho, (f, g, h)}(A) J_{\rho, (f, g, h)}(B). \end{aligned}$$

Similarly we have

$$\beta(f, g, h) |Tr[f(\rho)g(\rho)h(\rho)[A, B]]|^2 \leq I_{\rho, (f, g, h)}(B) J_{\rho, (f, g, h)}(A).$$

Hence by multiplying the above two inequalities, we have

$$\beta(f, g, h) |Tr[f(\rho)g(\rho)h(\rho)[A, B]]|^2 \leq U_{\rho, (f, g, h)}(A) U_{\rho, (f, g, h)}(B).$$

□

When $h(x) = 1$, we obtain the result given by Ko and Yoo [6].

Corollary 4.1 ([6]) *If (f, g) is CLI monotone pair, then the following inequality holds:*

$$U_{\rho, (f, g)}(A) U_{\rho, (f, g)}(B) \geq \beta(f, g) |Tr[f(\rho)g(\rho)[A, B]]|^2$$

for $A, B \in M_{n, sa}(\mathbb{C})$, where

$$\begin{aligned} I_{\rho, (f, g)}(A) &= \frac{1}{2} Tr[(i[f(\rho), A_0])(i[g(\rho), A_0])], \\ J_{\rho, (f, g)}(A) &= \frac{1}{2} Tr[\{f(\rho), A_0\}\{g(\rho), A_0\}], \\ U_{\rho, (f, g)}(A) &= \sqrt{I_{\rho, (f, g)} J_{\rho, (f, g)}}, \\ \beta(f, g) &= \min\left\{\frac{m}{(m+M)^2}, \frac{M}{(m+M)^2}\right\}. \end{aligned}$$

We also have the following corollary.

Corollary 4.2 *Let $f(x) = x^\alpha$ ($\alpha \geq 0$), $g(x) = x^\beta$ ($\beta \geq 0$), $h(x) = x^\gamma$ ($\gamma \geq 0$ or $\gamma \leq 0$).*

(1) *If $\alpha, \beta, \gamma \geq 0$ satisfy $0 < \alpha + \beta \leq \gamma$, then*

$$\beta(f, g, h) = \frac{\alpha\beta}{(\alpha + \beta + \gamma)^2}.$$

(2) *If $\alpha, \beta \geq 0, \gamma \leq 0$ satisfy $\alpha + \beta + \gamma > 0$, then*

$$\beta(f, g, h) = \frac{\alpha\beta}{(\alpha + \beta + \gamma)^2}.$$

Remark 4.1 When $\alpha, \beta \geq 0, \gamma < 0$ satisfy $\alpha + \beta + \gamma > 0$, we remark that $h(x)$ is not continuous function on $[0, 1]$ because

$$\lim_{x \rightarrow +0} h(x) = +\infty.$$

Then in this case by putting $\epsilon > 0$ such that ϵ is smaller than the minimal eigenvalue of ρ , we can assume that $h(x)$ is continuous on $[\epsilon, 1]$. Hence we obtain the same result as Corollary 4.2.

Remark 4.2 When $\gamma = 0$ in (2) of Corollary 4.2, we have the result in [15] (Theorem 2.3). And when $\alpha + \beta + \gamma = 1$ in Corollary 4.2, we have the result in [14] (Theorem 2.2). That is (1) implies $\alpha, \beta \geq 0, \alpha + \beta \leq \frac{1}{2}$ and (2) implies $\alpha, \beta \geq 0, \alpha + \beta \geq 1$.

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